IMO Shortlist 2000

Algebra

1. Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

2. Let a, b, c be positive integers satisfying the conditions $b > 2a$ and $c > 2b$. Show that there exists a real number λ with the property that all the three numbers $\lambda a, \lambda b, \lambda c$ have their fractional parts lying in the interval $\left(\frac{1}{2}, \frac{2}{3}\right]$.

3. Find all pairs of functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x + g(y)) = x \cdot f(y) - y \cdot f(x) + g(x)$ for all real x, y.

4. The function F is defined on the set of nonnegative integers and takes nonnegative integer values satisfying the following conditions: for every $n \geq 0$,

   (i) $F(4n) = F(2n) + F(n)$, (ii) $F(4n + 2) = F(4n) + 1$, (iii) $F(2n + 1) = F(2n) + 1$.

   Prove that for each positive integer m, the number of integers n with $0 \leq n < 2^m$ and $F(4n) = F(3n)$ is $F(2^{m+1})$.

5. Let $n \geq 2$ be a positive integer and λ a positive real number. Initially there are n fleas on a horizontal line, not all at the same point. We define a move as choosing two fleas at some points A and B, with A to the left of B, and letting the flea from A jump over the flea from B to the point C so that $\frac{BC}{AB} = \lambda$.

   Determine all values of λ such that, for any point M on the line and for any initial position of the n fleas, there exists a sequence of moves that will take them all to the position right of M.

6. A nonempty set A of real numbers is called a B_3-set if the conditions $a_1, a_2, a_3, a_4, a_5, a_6 \in A$ and $a_1 + a_2 + a_3 = a_4 + a_5 + a_6$ imply that the sequences (a_1, a_2, a_3) and (a_4, a_5, a_6) are identical up to a permutation. Let

$$A = \{a(0) = 0 < a(1) < a(2) < \ldots\}, B = \{b(0) = 0 < b(1) < b(2) < \ldots\}$$

   be infinite sequences of real numbers with $D(A) = D(B)$, where, for a set X of real numbers, $D(X)$ denotes the difference set $\{|x - y| | x, y \in X\}$. Prove that if A is a B_3-set, then $A = B$.

$\boxed{7}$ For a polynomial P of degree 2000 with distinct real coefficients let $M(P)$ be the set of all polynomials that can be produced from P by permutation of its coefficients. A polynomial P will be called n-**independent** if $P(n) = 0$ and we can get from any $Q \in M(P)$ a polynomial Q_1 such that $Q_1(n) = 0$ by interchanging at most one pair of coefficients of Q. Find all integers n for which n-independent polynomials exist.

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Combinatorics

$\boxed{1}$ A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card. A member of the audience draws two cards from two different boxes and announces the sum of numbers on those cards. Given this information, the magician locates the box from which no card has been drawn.

How many ways are there to put the cards in the three boxes so that the trick works?

$\boxed{2}$ A staircase-brick with 3 steps of width 2 is made of 12 unit cubes. Determine all integers n for which it is possible to build a cube of side n using such bricks.

$\boxed{3}$ Let $n \geq 4$ be a fixed positive integer. Given a set $S = \{P_1, P_2, \dots, P_n\}$ of n points in the plane such that no three are collinear and no four concyclic, let a_t, $1 \leq t \leq n$, be the number of circles $P_iP_jP_k$ that contain P_t in their interior, and let $m(S) = \sum_{i=1}^n a_i$. Prove that there exists a positive integer $f(n)$, depending only on n, such that the points of S are the vertices of a convex polygon if and only if $m(S) = f(n)$.

$\boxed{4}$ Let n and k be positive integers such that $\frac{1}{2}n < k \leq \frac{2}{3}n$. Find the least number m for which it is possible to place m pawns on m squares of an $n \times n$ chessboard so that no column or row contains a block of k adjacent unoccupied squares.

$\boxed{5}$ A number of n rectangles are drawn in the plane. Each rectangle has parallel sides and the sides of distinct rectangles lie on distinct lines. The rectangles divide the plane into a number of regions. For each region R let $v(R)$ be the number of vertices. Take the sum $\sum v(R)$ over the regions which have one or more vertices of the rectangles in their boundary. Show that this sum is less than $40n$.

$\boxed{6}$ Let p and q be relatively prime positive integers. A subset S of $\{0, 1, 2, \dots\}$ is called **ideal** if $0 \in S$ and for each element $n \in S$, the integers $n + p$ and $n + q$ belong to S. Determine the number of ideal subsets of $\{0, 1, 2, \dots\}$.

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Geometry

1. In the plane we are given two circles intersecting at X and Y. Prove that there exist four points with the following property:

   (P) For every circle touching the two given circles at A and B, and meeting the line XY at C and D, each of the lines AC, AD, BC, BD passes through one of these points.

2. Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

3. Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that

   $$OD + DH = OE + EH = OF + FH$$

   and the lines AD, BE, and CF are concurrent.

4. Let $A_1A_2\dots A_n$ be a convex polygon, $n \geq 4$. Prove that $A_1A_2\dots A_n$ is cyclic if and only if to each vertex A_j one can assign a pair (b_j, c_j) of real numbers, $j = 1, 2, \dots, n$, so that $A_iA_j = b_jc_ib_ic_j$ for all i, j with $1 \leq i < j \leq n$.

5. Let ABC be an acute-angled triangle, and let w be the circumcircle of triangle ABC.

   The tangent to the circle w at the point A meets the tangent to the circle w at C at the point B'. The line BB' intersects the line AC at E, and N is the midpoint of the segment BE.

   Similarly, the tangent to the circle w at the point B meets the tangent to the circle w at the point C at the point A'. The line AA' intersects the line BC at D, and M is the midpoint of the segment AD.

   a) Show that $\measuredangle ABM = \measuredangle BAN$. **b)** If $AB = 1$, determine the values of BC and AC for the triangles ABC which maximise $\measuredangle ABM$.

6. Let $ABCD$ be a convex quadrilateral. The perpendicular bisectors of its sides AB and CD meet at Y. Denote by X a point inside the quadrilateral $ABCD$ such that $\measuredangle ADX = \measuredangle BCX < 90^\circ$ and $\measuredangle DAX = \measuredangle CBX < 90^\circ$. Show that $\measuredangle AYB = 2 \cdot \measuredangle ADX$.

7. Ten gangsters are standing on a flat surface, and the distances between them are all distinct. At twelve oclock, when the church bells start chiming, each of them fatally shoots the one among the other nine gangsters who is the nearest. At least how many gangsters will be killed?

$\boxed{8}$ Let AH_1, BH_2, CH_3 be the altitudes of an acute angled triangle ABC. Its incircle touches the sides BC, AC and AB at T_1, T_2 and T_3 respectively. Consider the symmetric images of the lines H_1H_2, H_2H_3 and H_3H_1 with respect to the lines T_1T_2, T_2T_3 and T_3T_1. Prove that these images form a triangle whose vertices lie on the incircle of ABC.

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Number Theory

1. Determine all positive integers $n \geq 2$ that satisfy the following condition: for all a and b relatively prime to n we have $a \equiv b \pmod{n}$ if and only if $ab \equiv 1 \pmod{n}$.

2. For every positive integers n let $d(n)$ the number of all positive integers of n. Determine all positive integers n with the property: $d^3(n) = 4n$.

3. Does there exist a positive integer n such that n has exactly 2000 prime divisors and n divides $2^n + 1$?

4. Find all triplets of positive integers (a, m, n) such that $a^m + 1 \mid (a+1)^n$.

5. Prove that there exist infinitely many positive integers n such that $p = nr$, where p and r are respectively the semiperimeter and the inradius of a triangle with integer side lengths.

6. Show that the set of positive integers that cannot be represented as a sum of distinct perfect squares is finite.