IMO Shortlist 1995

Algebra

$\boxed{1}$ Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

$\boxed{2}$ Let a and b be non-negative integers such that $ab \geq c^2$, where c is an integer. Prove that there is a number n and integers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ such that

$$\sum_{i=1}^{n} x_i^2 = a, \sum_{i=1}^{n} y_i^2 = b, \text{ and } \sum_{i=1}^{n} x_i y_i = c.$$

$\boxed{3}$ Let n be an integer, $n \geq 3$. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $2 \leq a_i \leq 3$ for $i = 1, 2, \ldots, n$. If $s = a_1 + a_2 + \ldots + a_n$, prove that

$$\frac{a_1^2 + a_2^2 - a_3^2}{a_1 + a_2 - a_3} + \frac{a_2^2 + a_3^2 - a_4^2}{a_2 + a_3 - a_4} + \ldots + \frac{a_n^2 + a_1^2 - 2a^2}{a_n + a_1 - a_2} \leq 2s - 2n.$$

$\boxed{4}$ Find all of the positive real numbers like x, y, z, such that :

1.) $x + y + z = a + b + c$

2.) $4xyz = a^2x + b^2y + c^2z + abc$

Proposed to Gazeta Matematica in the 80s by VASILE CRTOAJE and then by Titu Andreescu to IMO 1995.

$\boxed{5}$ Let $\mathbb{R}$ be the set of real numbers. Does there exist a function $f : \mathbb{R} \mapsto \mathbb{R}$ which simultaneously satisfies the following three conditions?

(a) There is a positive number M such that $\forall x : -M \leq f(x) \leq M$. **(b)** The value of f(1) is 1. **(c)** If $x \neq 0$, then

$$f\left(x + \frac{1}{x^2}\right) = f(x) + \left[f\left(\frac{1}{x}\right)\right]^2$$

$\boxed{6}$ Let n be an integer,$n \geq 3$. Let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_i < x_{i+1}$ for $1 \leq i \leq n-1$. Prove that

$$\frac{n(n-1)}{2} \sum_{i<j} x_i x_j > \left(\sum_{i=1}^{n-1} (n-i) \cdot x_i\right) \cdot \left(\sum_{j=2}^{n} (j-1) \cdot x_j\right)$$

Geometry

1 Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

2 Let A, B and C be non-collinear points. Prove that there is a unique point X in the plane of ABC such that

$$XA^2 + XB^2 + AB^2 = XB^2 + XC^2 + BC^2 = XC^2 + XA^2 + CA^2.$$

3 The incircle of triangle $\triangle ABC$ touches the sides BC, CA, AB at D, E, F respectively. X is a point inside triangle of $\triangle ABC$ such that the incircle of triangle $\triangle XBC$ touches BC at D, and touches CX and XB at Y and Z respectively. Show that E, F, Z, Y are concyclic.

4 An acute triangle ABC is given. Points A_1 and A_2 are taken on the side BC (with A_2 between A_1 and C), B_1 and B_2 on the side AC (with B_2 between B_1 and A), and C_1 and C_2 on the side AB (with C_2 between C_1 and B) so that

$$\angle AA_1A_2 = \angle AA_2A_1 = \angle BB_1B_2 = \angle BB_2B_1 = \angle CC_1C_2 = \angle CC_2C_1.$$

The lines AA_1, BB_1, and CC_1 bound a triangle, and the lines AA_2, BB_2, and CC_2 bound a second triangle. Prove that all six vertices of these two triangles lie on a single circle.

5 Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\angle BCD = \angle EFA = \frac{\pi}{3}$. Suppose G and H are points in the interior of the hexagon such that $\angle AGB = \angle DHE = \frac{2\pi}{3}$. Prove that $AG + GB + GH + DH + HE \geq CF$.

6 Let $A_1A_2A_3A_4$ be a tetrahedron, G its centroid, and A_1', A_2', A_3', and A_4' the points where the circumsphere of $A_1A_2A_3A_4$ intersects GA_1, GA_2, GA_3, and GA_4, respectively. Prove that

$$GA_1 \cdot GA_2 \cdot GA_3 \cdot GA.4 \leq GA_1' \cdot GA_2' \cdot GA_3' \cdot GA_4'$$

and

$$\frac{1}{GA_1'} + \frac{1}{GA_2'} + \frac{1}{GA_3'} + \frac{1}{GA_4'} \leq \frac{1}{GA_1} + \frac{1}{GA_2} + \frac{1}{GA_3} + \frac{1}{GA_4}.$$

$\boxed{7}$ Let ABCD be a convex quadrilateral and O a point inside it. Let the parallels to the lines BC, AB, DA, CD through the point O meet the sides AB, BC, CD, DA of the quadrilateral ABCD at the points E, F, G, H, respectively. Then, prove that $\sqrt{|AHOE|} + \sqrt{|CFOG|} \leq \sqrt{|ABCD|}$, where $|P_1 P_2 ... P_n|$ is an abbreviation for the non-directed area of an arbitrary polygon $P_1 P_2 ... P_n$.

$\boxed{8}$ Suppose that $ABCD$ is a cyclic quadrilateral. Let $E = AC \cap BD$ and $F = AB \cap CD$. Denote by H_1 and H_2 the orthocenters of triangles EAD and EBC, respectively. Prove that the points F, H_1, H_2 are collinear.

Original formulation:

Let ABC be a triangle. A circle passing through B and C intersects the sides AB and AC again at C' and B', respectively. Prove that BB', CC' and HH' are concurrent, where H and H' are the orthocentres of triangles ABC and $AB'C'$ respectively.

NT, Combs

1 Let k be a positive integer. Show that there are infinitely many perfect squares of the form $n \cdot 2^k - 7$ where n is a positive integer.

2 Let $\mathbb{Z}$ denote the set of all integers. Prove that for any integers A and B, one can find an integer C for which $M_1 = \{x^2 + Ax + B : x \in \mathbb{Z}\}$ and $M_2 = 2x^2 + 2x + C : x \in \mathbb{Z}$ do not intersect.

3 Determine all integers $n > 3$ for which there exist n points $A_1, \cdots, A_n$ in the plane, no three collinear, and real numbers $r_1, \cdots, r_n$ such that for $1 \leq i < j < k \leq n$, the area of $\triangle A_i A_j A_k$ is $r_i + r_j + r_k$.

4 Find all x, y and z in positive integer: $z + y^2 + x^3 = xyz$ and $x = \gcd(y, z)$.

5 At a meeting of $12k$ people, each person exchanges greetings with exactly $3k + 6$ others. For any two people, the number who exchange greetings with both is the same. How many people are at the meeting?

6 Let p be an odd prime number. How many p-element subsets A of $\{1, 2, \cdots\ 2p\}$ are there, the sum of whose elements is divisible by p?

7 Does there exist an integer $n > 1$ which satisfies the following condition? The set of positive integers can be partitioned into n nonempty subsets, such that an arbitrary sum of $n - 1$ integers, one taken from each of any $n - 1$ of the subsets, lies in the remaining subset.

8 Let p be an odd prime. Determine positive integers x and y for which $x \leq y$ and $\sqrt{2p} - \sqrt{x} - \sqrt{y}$ is non-negative and as small as possible.

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Sequences

$\boxed{1}$ Does there exist a sequence $F(1), F(2), F(3), \ldots$ of non-negative integers that simultaneously satisfies the following three conditions?

(a) Each of the integers $0, 1, 2, \ldots$ occurs in the sequence. **(b)** Each positive integer occurs in the sequence infinitely often. **(c)** For any $n \geq 2$,

$$F(F(n^{163})) = F(F(n)) + F(F(361)).$$

$\boxed{2}$ Find the maximum value of x_0 for which there exists a sequence $x_0, x_1 \cdots, x_{1995}$ of positive reals with $x_0 = x_{1995}$, such that

$$x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i},$$

for all $i = 1, \cdots, 1995$.

$\boxed{3}$ For an integer $x \geq 1$, let $p(x)$ be the least prime that does not divide x, and define $q(x)$ to be the product of all primes less than $p(x)$. In particular, $p(1) = 2$. For x having $p(x) = 2$, define $q(x) = 1$. Consider the sequence $x_0, x_1, x_2, \ldots$ defined by $x_0 = 1$ and

$$x_{n+1} = \frac{x_n p(x_n)}{q(x_n)}$$

for $n \geq 0$. Find all n such that $x^n = 1995$.

$\boxed{4}$ Suppose that $x_1, x_2, x_3, \ldots$ are positive real numbers for which

$$x_n^n = \sum_{j=0}^{n-1} x_n^j$$

for $n = 1, 2, 3, \ldots$ Prove that $\forall n$,

$$2 - \frac{1}{2^{n-1}} \leq x_n < 2 - \frac{1}{2^n}.$$

$\boxed{5}$ For positive integers n, the numbers $f(n)$ are defined inductively as follows: $f(1) = 1$, and for every positive integer n, $f(n+1)$ is the greatest integer m such that there is an arithmetic progression of positive integers $a_1 < a_2 < \ldots < a_m = n$ for which

$$f(a_1) = f(a_2) = \ldots = f(a_m).$$

Prove that there are positive integers a and b such that $f(an + b) = n + 2$ for every positive integer n.

$\boxed{6}$ Let $\mathbb{N}$ denote the set of all positive integers. Prove that there exists a unique function $f : \mathbb{N} \mapsto \mathbb{N}$ satisfying

$$f(m + f(n)) = n + f(m + 95)$$

for all m and n in $\mathbb{N}$. What is the value of $\sum_{k=1}^{19} f(k)$?