# IMO Shortlist 1991

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$\boxed{1}$ Given a point $P$ inside a triangle $\triangle ABC$. Let $D$, $E$, $F$ be the orthogonal projections of the point $P$ on the sides $BC$, $CA$, $AB$, respectively. Let the orthogonal projections of the point $A$ on the lines $BP$ and $CP$ be $M$ and $N$, respectively. Prove that the lines $ME$, $NF$, $BC$ are concurrent.

*Original formulation:*

Let $ABC$ be any triangle and $P$ any point in its interior. Let $P_1, P_2$ be the feet of the perpendiculars from $P$ to the two sides $AC$ and $BC$. Draw $AP$ and $BP$, and from $C$ drop perpendiculars to $AP$ and $BP$. Let $Q_1$ and $Q_2$ be the feet of these perpendiculars. Prove that the lines $Q_1P_2, Q_2P_1$, and $AB$ are concurrent.

$\boxed{2}$ $ABC$ is an acute-angled triangle. $M$ is the midpoint of $BC$ and $P$ is the point on $AM$ such that $MB = MP$. $H$ is the foot of the perpendicular from $P$ to $BC$. The lines through $H$ perpendicular to $PB$, $PC$ meet $AB, AC$ respectively at $Q, R$. Show that $BC$ is tangent to the circle through $Q, H, R$ at $H$.

*Original Formulation:*

For an acute triangle $ABC, M$ is the midpoint of the segment $BC, P$ is a point on the segment $AM$ such that $PM = BM, H$ is the foot of the perpendicular line from $P$ to $BC, Q$ is the point of intersection of segment $AB$ and the line passing through $H$ that is perpendicular to $PB$, and finally, $R$ is the point of intersection of the segment $AC$ and the line passing through $H$ that is perpendicular to $PC$. Show that the circumcircle of $QHR$ is tangent to the side $BC$ at point $H$.

$\boxed{3}$ Let $S$ be any point on the circumscribed circle of $PQR$. Then the feet of the perpendiculars from S to the three sides of the triangle lie on the same straight line. Denote this line by $l(S, PQR)$. Suppose that the hexagon $ABCDEF$ is inscribed in a circle. Show that the four lines $l(A, BDF)$, $l(B, ACE)$, $l(D, ABF)$, and $l(E, ABC)$ intersect at one point if and only if $CDEF$ is a rectangle.

$\boxed{4}$ Let $ABC$ be a triangle and $P$ an interior point of $ABC$. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to $30^\circ$.

$\boxed{5}$ In the triangle $ABC$, with $\angle A = 60^\circ$, a parallel $IF$ to $AC$ is drawn through the incenter $I$ of the triangle, where $F$ lies on the side $AB$. The point $P$ on the side $BC$ is such that $3BP = BC$. Show that $\angle BFP = \frac{\angle B}{2}$.

$\boxed{7}$ $ABCD$ is a terahedron: $AD + BD = AC + BC$, $BD + CD = BA + CA$, $CD + AD = CB + AB$, $M, N, P$ are the mid points of $BC, CA, AB$. $OA = OB = OC = OD$. Prove that $\angle MOP = \angle NOP = \angle NOM$.

8 $S$ be a set of $n$ points in the plane. No three points of $S$ are collinear. Prove that there exists a set $P$ containing $2n-5$ points satisfying the following condition: In the interior of every triangle whose three vertices are elements of $S$ lies a point that is an element of $P$.

9 In the plane we are given a set $E$ of 1991 points, and certain pairs of these points are joined with a path. We suppose that for every point of $E$, there exist at least 1593 other points of $E$ to which it is joined by a path. Show that there exist six points of $E$ every pair of which are joined by a path.

*Alternative version:* Is it possible to find a set $E$ of 1991 points in the plane and paths joining certain pairs of the points in $E$ such that every point of $E$ is joined with a path to at least 1592 other points of $E$, and in every subset of six points of $E$ there exist at least two points that are not joined?

10 Suppose $G$ is a connected graph with $k$ edges. Prove that it is possible to label the edges $1, 2, \ldots, k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.

[hide="Graph-Definition"]A **graph** consists of a set of points, called vertices, together with a set of edges joining certain pairs of distinct vertices. Each pair of vertices $u, v$ belongs to at most one edge. The graph $G$ is connected if for each pair of distinct vertices $x, y$ there is some sequence of vertices $x = v_0, v_1, v_2, \cdots, v_m = y$ such that each pair $v_i, v_{i+1}$ $(0 \leq i < m)$ is joined by an edge of $G$.

11 Prove that $\sum_{k=0}^{995} \frac{(-1)^k}{1991-k}\binom{1991-k}{k} = \frac{1}{1991}$

12 Let $S = \{1, 2, 3, \cdots, 280\}$. Find the smallest integer $n$ such that each $n$-element subset of $S$ contains five numbers which are pairwise relatively prime.

13 Given any integer $n \geq 2$, assume that the integers $a_1, a_2, \ldots, a_n$ are not divisible by $n$ and, moreover, that $n$ does not divide $\sum_{i=1}^n a_i$. Prove that there exist at least $n$ different sequences $(e_1, e_2, \ldots, e_n)$ consisting of zeros or ones such $\sum_{i=1}^n e_i \cdot a_i$ is divisible by $n$.

14 Let $a, b, c$ be integers and $p$ an odd prime number. Prove that if $f(x) = ax^2 + bx + c$ is a perfect square for $2p-1$ consecutive integer values of $x$, then $p$ divides $b^2 - 4ac$.

15 Let $a_n$ be the last nonzero digit in the decimal representation of the number $n!$. Does the sequence $a_1, a_2, \ldots, a_n, \ldots$ become periodic after a finite number of terms?

16 Let $n > 6$ be an integer and $a_1, a_2, \cdots, a_k$ be all the natural numbers less than $n$ and relatively prime to $n$. If

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0,$$

prove that $n$ must be either a prime number or a power of 2.

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$\boxed{17}$ Find all positive integer solutions $x, y, z$ of the equation $3^x + 4^y = 5^z$.

$\boxed{18}$ Find the highest degree $k$ of 1991 for which $1991^k$ divides the number

$$1990^{1991^{1992}} + 1992^{1991^{1990}}.$$

$\boxed{19}$ Let $\alpha$ be a rational number with $0 < \alpha < 1$ and $\cos(3\pi\alpha) + 2\cos(2\pi\alpha) = 0$. Prove that $\alpha = \frac{2}{3}$.

$\boxed{20}$ Let $\alpha$ be the positive root of the equation $x^2 = 1991x + 1$. For natural numbers $m$ and $n$ define

$$m * n = mn + \lfloor \alpha m \rfloor \lfloor \alpha n \rfloor.$$

Prove that for all natural numbers $p$, $q$, and $r$,

$$(p * q) * r = p * (q * r).$$

$\boxed{21}$ Let $f(x)$ be a monic polynomial of degree 1991 with integer coefficients. Define $g(x) = f^2(x) - 9$. Show that the number of distinct integer solutions of $g(x) = 0$ cannot exceed 1995.

$\boxed{22}$ Real constants $a, b, c$ are such that there is exactly one square all of whose vertices lie on the cubic curve $y = x^3 + ax^2 + bx + c$. Prove that the square has sides of length $\sqrt[4]{72}$.

$\boxed{23}$ Let $f$ and $g$ be two integer-valued functions defined on the set of all integers such that

*(a)* $f(m + f(f(n))) = -f(f(m+1) - n$ for all integers $m$ and $n$; *(b)* $g$ is a polynomial function with integer coefficients and g(n) $= g(f(n))$ $\forall n \in \mathbb{Z}$.

$\boxed{24}$ An odd integer $n \geq 3$ is said to be nice if and only if there is at least one permutation $a_1, \cdots, a_n$ of $1, \cdots, n$ such that the $n$ sums $a_1 - a_2 + a_3 - \cdots - a_{n-1} + a_n$, $a_2 - a_3 + a_3 - \cdots - a_n + a_1$, $a_3 - a_4 + a_5 - \cdots - a_1 + a_2$, $\cdots$, $a_n - a_1 + a_2 - \cdots - a_{n-2} + a_{n-1}$ are all positive. Determine the set of all 'nice' integers.

$\boxed{25}$ Suppose that $n \geq 2$ and $x_1, x_2, \ldots, x_n$ are real numbers between 0 and 1 (inclusive). Prove that for some index $i$ between 1 and $n-1$ the inequality

$$x_i(1 - x_{i+1}) \geq \frac{1}{4} x_1(1 - x_n)$$

$\boxed{26}$ Let $n \geq 2, n \in \mathbb{N}$ and let $p, a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathbb{R}$ satisfying $\frac{1}{2} \leq p \leq 1$, $0 \leq a_i$, $0 \leq b_i \leq p$, $i = 1, \ldots, n$, and

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i.$$

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Prove the inequality:

$$\sum_{i=1}^{n} b_i \prod_{j=1, j\neq i}^{n} a_j \leq \frac{p}{(n-1)^{n-1}}.$$

27 Determine the maximum value of the sum

$$\sum_{i<j} x_i x_j (x_i + x_j)$$

over all $n-$tuples $(x_1, \ldots, x_n)$, satisfying $x_i \geq 0$ and $\sum_{i=1}^{n} x_i = 1$.

28 An infinite sequence $x_0, x_1, x_2, \ldots$ of real numbers is said to be **bounded** if there is a constant $C$ such that $|x_i| \leq C$ for every $i \geq 0$. Given any real number $a > 1$, construct a bounded infinite sequence $x_0, x_1, x_2, \ldots$ such that

$$|x_i - x_j||i - j|^a \geq 1$$

for every pair of distinct nonnegative integers $i, j$.

29 We call a set $S$ on the real line $\mathbb{R}$ *superinvariant* if for any stretching $A$ of the set by the transformation taking $x$ to $A(x) = x_0 + a(x - x_0), a > 0$ there exists a translation $B$, $B(x) = x + b$, such that the images of $S$ under $A$ and $B$ agree; i.e., for any $x \in S$ there is a $y \in S$ such that $A(x) = B(y)$ and for any $t \in S$ there is a $u \in S$ such that $B(t) = A(u)$. Determine all *superinvariant* sets.

30 Two students $A$ and $B$ are playing the following game: Each of them writes down on a sheet of paper a positive integer and gives the sheet to the referee. The referee writes down on a blackboard two integers, one of which is the sum of the integers written by the players. After that, the referee asks student $A$ : Can you tell the integer written by the other student? If A answers no, the referee puts the same question to student $B$. If $B$ answers no, the referee puts the question back to $A$, and so on. Assume that both students are intelligent and truthful. Prove that after a finite number of questions, one of the students will answer yes.